

NEAR PERFECT MATCHINGS IN k -UNIFORM HYPERGRAPHS

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ABSTRACT. Let H be a k -uniform hypergraph on n vertices where n is a sufficiently large integer not divisible by k . We prove that if the minimum $(k-1)$ -degree of H is at least $\lfloor n/k \rfloor$, then H contains a matching with $\lfloor n/k \rfloor$ edges. This confirms a conjecture of Rödl, Ruciński and Szemerédi [13], who proved that minimum $(k-1)$ -degree $n/k + O(\log n)$ suffices. More generally, we show that H contains a matching of size d if its minimum codegree is $d < n/k$, which is also best possible.

1. INTRODUCTION

Given $k \geq 2$, a k -uniform hypergraph (in short, k -graph) consists of a vertex set $V(H)$ and an edge set $E(H) \subseteq \binom{V(H)}{k}$, where every edge is a k -element subset of $V(H)$. A *matching* in H is a collection of vertex-disjoint edges of H . A *perfect matching* M in H is a matching that covers all vertices of H . Clearly a perfect matching in H exists only if k divides $|V(H)|$. When k does not divide $n = |V(H)|$, we call a matching M in H a *near perfect matching* if $|M| = \lfloor n/k \rfloor$.

Given a k -graph H with a set S of d vertices (where $1 \leq d \leq k-1$) we define $\deg_H(S)$ to be the number of edges containing S (the subscript H is omitted if it is clear from the context). The *minimum d -degree* $\delta_d(H)$ of H is the minimum of $\deg_H(S)$ over all d -vertex sets S in H . We refer to $\delta_{k-1}(H)$ as the *minimum codegree* of H .

Over the last few years there has been a strong focus in establishing minimum d -degree thresholds that force a perfect matching in a k -graph [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14]. In particular, Rödl, Ruciński and Szemerédi [13] determined the minimum codegree threshold that ensures a perfect matching in a k -graph on n vertices for all $k \geq 3$ and sufficiently large $n \in k\mathbb{N}$. The threshold is $\frac{n}{2} - k + C$, where $C \in \{3/2, 2, 5/2, 3\}$ depends on the values of n and k . In contrast, they proved that the minimum codegree threshold that ensures a near perfect matching in a k -graph on $n \notin k\mathbb{N}$ vertices is between $\lfloor \frac{n}{k} \rfloor$ and $\frac{n}{k} + O(\log n)$. It is conjectured, in [13] and [10, Problem 3.3], that this threshold is $\lfloor \frac{n}{k} \rfloor$. In this note we verify this conjecture.

Theorem 1.1. *For any integer $k \geq 3$, let n be a sufficiently large integer which is not divisible by k . Suppose H is a k -uniform hypergraph on n vertices with $\delta_{k-1}(H) \geq \lfloor \frac{n}{k} \rfloor$. Then H contains a matching of size $\lfloor \frac{n}{k} \rfloor$.*

It is also natural to ask for the minimum codegree threshold for the *matching number* of k -graphs, namely, the size of a maximum matching. The following

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theorem [13, Fact 2.1] is obtained by a greedy algorithm. Let $\nu(H)$ be the size of a maximum matching in H .

Theorem 1.2. [13] *Let $n \geq k \geq 2$. For every k -uniform hypergraph H on n vertices,*

$$\nu(H) \geq \delta_{k-1}(H) \text{ if } \delta_{k-1}(H) \leq \left\lfloor \frac{n}{k} \right\rfloor - k + 2.$$

Note that for $n \in k\mathbb{N}$ and $\frac{n}{k} \leq \delta_{k-1}(H) \leq \frac{n}{2} - k$, H may not contain a perfect matching, namely, a matching of size $\frac{n}{k}$ (see [13]). So the only open cases are when $\left\lfloor \frac{n}{k} \right\rfloor - k + 3 \leq \delta_{k-1}(H) < \frac{n}{k}$. In this note, we close this gap for large n .

Corollary 1.3. *For any integer $k \geq 3$, let n be a sufficiently large integer. For every k -uniform hypergraph H on n vertices,*

$$\nu(H) \geq \delta_{k-1}(H) \text{ if } \delta_{k-1}(H) < \frac{n}{k}.$$

Proof. Let $\delta_{k-1}(H) = \left\lfloor \frac{n}{k} \right\rfloor - c$. We only prove Corollary 1.3 in the cases when $1 \leq c \leq k - 3$, since Theorem 1.2 covers the cases when $c \geq k - 2$ and Theorem 1.1 covers the case when $\delta_{k-1}(H) = \left\lfloor \frac{n}{k} \right\rfloor < \frac{n}{k}$. Let $r \equiv n \pmod k$ such that $0 \leq r \leq k - 1$. Note that $\left\lfloor \frac{n}{k} \right\rfloor = \left\lfloor \frac{n+c}{k} \right\rfloor$ if $r + c < k$ and $\left\lfloor \frac{n}{k} \right\rfloor + 1 = \left\lfloor \frac{n+c+1}{k} \right\rfloor$ otherwise. For the first case, we add c vertices to H and get H' such that H' contains all edges of H and all k -sets containing any of these new vertices. Note that H' has $n + c$ vertices and $\delta_{k-1}(H') = \left\lfloor \frac{n+c}{k} \right\rfloor$. Moreover, k does not divide $n + c$ since $1 \leq r + c < k$. We apply Theorem 1.1 on H' and get a near perfect matching M of H' . Deleting up to c edges from M that contain the new vertices, we get a matching in H of size $\left\lfloor \frac{n}{k} \right\rfloor - c$.

In the second case, we add $c + 1$ vertices to H and get H' such that H' contains all edges of H and all k -sets containing any of these new vertices. Note that H' has $n + c + 1$ vertices and $\delta_{k-1}(H') = \left\lfloor \frac{n}{k} \right\rfloor + 1 = \left\lfloor \frac{n+c+1}{k} \right\rfloor$. Moreover, k does not divide $n + c + 1$ since $k + 1 \leq r + c + 1 \leq 2k - 3$. Similarly we apply Theorem 1.1 on H' and get a near perfect matching M of H' . Deleting up to $c + 1$ edges from M that contain the new vertices, we get a matching in H of size $\left\lfloor \frac{n}{k} \right\rfloor + 1 - (c + 1) = \left\lfloor \frac{n}{k} \right\rfloor - c$. \square

It is easy to see that Theorem 1.1 and Corollary 1.3 are best possible. For an integer $0 \leq d < \frac{n}{k}$, let H be a k -graph with a partition $A \cup B$ of the vertex set $V(H)$ such that $|A| = d$ and $E(H)$ consists of all k -tuples that intersect A . Since every edge intersects A , we have $\nu(H) = \delta_{k-1}(H) = |A| = d$.

Let us describe this interesting phenomenon by the following dynamic process. Consider a k -graph H on n vertices with $E(H) = \emptyset$ at the beginning and add edges to $E(H)$ gradually. Corollary 1.3 says $\nu(H) \geq \delta_{k-1}(H)$ when $\delta_{k-1}(H) < \frac{n}{k}$. In order to guarantee a perfect matching, $\delta_{k-1}(H)$ needs to be about $n/2$ [13].

As a typical approach to obtain exact results, our proof of Theorem 1.1 consists of an *extremal case* and a *nonextremal case*. We say that H is γ -*extremal* if $V(H)$ contains an independent subset B of order at least $(1 - \gamma)\frac{k-1}{k}n$.

Theorem 1.4 (Nonextremal case). *For any integer $k \geq 3$ and constant $\gamma > 0$, there is an integer n_0 such that the following holds. Let $n \geq n_0$ be an integer not divisible by k and let H be an n -vertex k -graph with $\delta_{k-1}(H) \geq \frac{n}{k} - \gamma n$. If H is not $5k\gamma$ -extremal, then H contains a near perfect matching.*

Theorem 1.5 (Extremal case). *For any integer $k \geq 3$, there exist an $\epsilon > 0$ and an integer n_1 such that the following holds. Let $n \geq n_1$ be an integer not divisible by k and let H be an n -vertex k -graph with $\delta_{k-1}(H) \geq \lfloor \frac{n}{k} \rfloor$. If H is ϵ -extremal, then H contains a near perfect matching.*

Theorem 1.1 follows from Theorem 1.4 and Theorem 1.5 immediately.

We prove Theorem 1.4 by the absorbing method, initiated by Rödl, Ruciński and Szemerédi [11]. Given a set S of $k+1$ vertices, we call an edge $e \in E(H)$ disjoint from S *S -absorbing* if there are two disjoint edges e_1 and e_2 in $E(H)$ such that $|e_1 \cap S| = k-1$, $|e_1 \cap e| = 1$, $|e_2 \cap S| = 2$, and $|e_2 \cap e| = k-2$. Note that this is not the absorbing in the usual sense because $e_1 \cup e_2$ misses one vertex of $S \cup e$. Let us explain how such absorbing works. Let S be a $(k+1)$ -set and M be a matching, where $V(M) \cap S = \emptyset$, which contains an S -absorbing edge e . Then M can “absorb” S by replacing e in M by e_1 and e_2 (one vertex of e becomes uncovered). The following absorbing lemma was proved in [13, Fact 2.3] with the conclusion that *the number of S -absorbing edges in M is at least $k-2$* . However, its proof shows that $k-2$ can be replaced by any constant. Note that we do not require that k does not divide n in Lemma 1.6 and Lemma 1.7.

Lemma 1.6. [13, Absorbing lemma] *For all $c, \gamma > 0$ there exist $C > 0$ and n_2 such that if H is a k -graph with $n \geq n_2$ vertices and $\delta_{k-1}(H) \geq cn$, then there exists a matching M' in H of size $|M'| \leq C \log n$ and such that for every $(k+1)$ -tuple S of vertices of H , the number of S -absorbing edges in M' is at least k/γ .*

We also need the following lemma, which provides a matching that covers all but a constant number of vertices when H is not extremal.

Lemma 1.7 (Almost perfect matching). *For any integer $k \geq 3$ and constant $\gamma > 0$ the following holds. Let H be an n -vertex k -graph such that n is sufficiently large and $\delta_{k-1}(H) \geq \frac{n}{k} - \gamma n$. If H is not $2k\gamma$ -extremal, then H contains a matching that covers all but at most k^2/γ vertices.*

Now let us compare our proof with the proof in [13], which showed that $\delta_{k-1}(H) \geq \frac{n}{k} + O(\log n)$ guarantees a near perfect matching. In [13], the authors first build an absorbing matching of size $C \log n$ and then apply Theorem 1.2 in the remaining k -graph. Finally, they absorb the leftover vertices and get the near perfect matching. In our proof, instead of Theorem 1.2, we apply Lemma 1.7 after building the absorbing matching. Lemma 1.7 only requires a weaker degree condition $\delta_{k-1}(H) \geq \frac{n}{k} - \gamma n$ and the condition that H is not extremal. We then handle the extremal case separately.

2. PROOF OF THEOREM 1.4

In this section we prove Theorem 1.4 with the help of Lemma 1.6 and Lemma 1.7.

Proof of Lemma 1.7. Let $M = \{e_1, e_2, \dots, e_m\}$ be a maximum matching of size m in H . Let V' be the set of vertices covered by M and let U be the set of vertices which are not covered by M . We assume that H is not $2k\gamma$ -extremal and $|U| > k^2/\gamma$. Note that U is an independent set by the maximality of M . We arbitrarily partition all but at most $k-2$ vertices of U as disjoint $(k-1)$ -sets A_1, \dots, A_t where $t = \lfloor \frac{|U|}{k-1} \rfloor > \frac{k}{\gamma}$.

Let D be the set of vertices $v \in V'$ such that $\{v\} \cup A_i \in E(H)$ for at least k sets A_i , $i \in [t]$. We claim that $|e_i \cap D| \leq 1$ for any $i \in [m]$. Otherwise, assume that $x, y \in e_i \cap D$. By the definition of D , we can pick A_i, A_j for some distinct $i, j \in [t]$ such that $\{x\} \cup A_i \in E(H)$ and $\{y\} \cup A_j \in E(H)$. We obtain a matching of size $m+1$ by replacing e_i in M by $\{x\} \cup A_i$ and $\{y\} \cup A_j$, contradicting the maximality of M .

Next we show that $|D| \geq (\frac{1}{k} - 2\gamma)n$. By the minimum degree condition, we have

$$t \left(\frac{1}{k} - \gamma \right) n \leq \sum_{i=1}^t \deg(A_i) \leq |D|t + n \cdot k,$$

where we use the fact that U is an independent set. So we get

$$|D| \geq \left(\frac{1}{k} - \gamma \right) n - \frac{nk}{t} > \left(\frac{1}{k} - 2\gamma \right) n,$$

where we use $t > k/\gamma$.

Let $V_D := \bigcup \{e_i, e_i \cap D \neq \emptyset\}$. Note that $|V_D \setminus D| = (k-1)|D| \geq (k-1)(\frac{1}{k} - 2\gamma)n$. Since H is not $2k\gamma$ -extremal, $H[V_D \setminus D]$ contains at least one edge, denoted by e_0 . We assume that e_0 intersects e_{i_1}, \dots, e_{i_l} in M for some $2 \leq l \leq k$. Suppose $\{v_{i_j}\} = e_{i_j} \cap D$ for all $j \in [l]$. By the definition of D , we can greedily pick A_{i_1}, \dots, A_{i_l} such that $\{v_{i_j}\} \cup A_{i_j} \in E(H)$ for all $j \in [l]$. Let M'' be the matching obtained from replacing the edges e_{i_1}, \dots, e_{i_l} by e_0 and $\{v_{i_j}\} \cup A_{i_j}$ for $j \in [l]$. Thus, M'' has $m+1$ edges, contradicting the maximality of M . \square

Now we prove Theorem 1.4.

Proof of Theorem 1.4. Suppose H is a k -graph on $n \notin k\mathbb{N}$ vertices with $\delta_{k-1}(H) \geq n/k - \gamma n$ and H is not $5k\gamma$ -extremal. In particular, $\gamma < \frac{1}{5k}$. Since $\delta_{k-1}(H) \geq \frac{n}{2k}$, we first apply Lemma 1.6 on H with $c = \frac{1}{2k}$ and find the absorbing matching M' of size at most $C \log n$ such that for every set S of $k+1$ vertices of H , the number of S -absorbing edges in M' is at least k/γ .

Let $H' = H[V(H) \setminus V(M')]$ and $n' = |V(H')|$. Note that $\delta_{k-1}(H') \geq \delta_{k-1}(H) - kC \log n > (\frac{1}{k} - 2\gamma)n'$. If H' is $4k\gamma$ -extremal, namely, $V(H')$ contains an independent set B of order at least $(1 - 4k\gamma)^{\frac{k-1}{k}} n'$, then since

$$(1 - 4k\gamma)^{\frac{k-1}{k}} n' \geq (1 - 5k\gamma)^{\frac{k-1}{k}} n,$$

we get that H is $5k\gamma$ -extremal, a contradiction. Thus, H' is not $4k\gamma$ -extremal and we can apply Lemma 1.7 on H' with parameter 2γ and get a matching M'' in H' that covers all but at most $k^2/(2\gamma)$ vertices. Since for every $(k+1)$ -tuple S in $V(H)$, the number of S -absorbing edges in M' is at least k/γ , we can repeatedly absorb the leftover vertices (at most $k/(2\gamma)$ times, each time the number of leftover vertices is reduced by k) until the number of leftover vertices is at most k (strictly less than k by the assumption). Let \tilde{M} denote the absorbing matching after the absorption. Then $\tilde{M} \cup M''$ is the desired near perfect matching in H . \square

3. PROOF OF THEOREM 1.5

We prove Theorem 1.5 in this section. We use the following result of Pikhurko [9], stated here in a less general form.

Theorem 3.1. [9, Theorem 3] *Let H be a k -partite k -graph with k -partition $V(H) = V_1 \cup V_2 \cup \dots \cup V_k$ such that $|V_i| = m$ for all $i \in [k]$. Let $\delta_{\{1\}}(H) = \min\{|N(v_1)| : v_1 \in V_1\}$ and*

$$\delta_{[k] \setminus \{1\}}(H) = \min\{|N(v_2, \dots, v_k)| : v_i \in V_i \text{ for every } 2 \leq i \leq k\}.$$

For sufficiently large integer m , if

$$\delta_{\{1\}}(H)m + \delta_{[k] \setminus \{1\}}(H)m^{k-1} \geq \frac{3}{2}m^k,$$

then H contains a perfect matching.

Proof of Theorem 1.5. Fix a sufficiently small $\epsilon > 0$. Suppose n is sufficiently large and not divisible by k . Let H be a k -graph on n vertices satisfying $\delta_{k-1}(H) \geq \lfloor \frac{n}{k} \rfloor$. Assume that H is ϵ -extremal, namely, there is an independent set $S \subseteq V(H)$ with $|S| \geq (1 - \epsilon)\frac{k-1}{k}n$.

We partition $V(H)$ as follows. Let $\alpha = \epsilon^{1/2}$. Let C be a maximum independent set of $V(H)$. Define

$$(3.1) \quad A = \left\{ x \in V \setminus C : \deg(x, C) \geq (1 - \alpha) \binom{|C|}{k-1} \right\},$$

and $B = V \setminus (A \cup C)$. We first observe the following bounds of $|A|, |B|, |C|$.

Proposition 3.2. $|A| \geq \lfloor \frac{n}{k} \rfloor - \alpha n$, $|B| \leq \alpha n$, and $(1 - \epsilon)\frac{(k-1)n}{k} \leq |C| \leq \lceil \frac{(k-1)n}{k} \rceil$.

Proof. The lower bound for $|C|$ follows from our hypothesis immediately. For any $S \subseteq C$ of order $k-1$, we have $N(S) \subseteq A \cup B$. By the minimum degree condition, we have

$$(3.2) \quad \left\lfloor \frac{n}{k} \right\rfloor \leq |N(S)| \leq |A| + |B| = n - |C| \leq \frac{n}{k} + \epsilon \frac{(k-1)n}{k},$$

which gives the upper bound for $|C|$. By the definitions of A and B , we have

$$\left\lfloor \frac{n}{k} \right\rfloor \binom{|C|}{k-1} \leq e((A \cup B)C^{k-1}) \leq (1 - \alpha) \binom{|C|}{k-1} |B| + \binom{|C|}{k-1} |A|,$$

where $e((A \cup B)C^{k-1})$ denotes the number of edges that contains $k-1$ vertices in C and one vertex in $A \cup B$. Thus, we get $\lfloor \frac{n}{k} \rfloor \leq |A| + |B| - \alpha|B|$, which gives that $\alpha|B| \leq |A| + |B| - \lfloor \frac{n}{k} \rfloor \leq \epsilon n$ by (3.2). So $|B| \leq \alpha n$ and $|A| \geq \lfloor \frac{n}{k} \rfloor - |B| \geq \lfloor \frac{n}{k} \rfloor - \alpha n$. \square

We will build four disjoint matchings M_1, M_2, M_3 , and M_4 in H , whose union gives the desired near perfect matching in H . Let $r \equiv n \pmod k$ and $1 \leq r \leq k-1$. Note that $\lfloor \frac{n}{k} \rfloor = \frac{n-r}{k}$. For $i \in [3]$, let $A_i = A \setminus V(\cup_{j \in [i]} M_j)$ and $C_i = C \setminus V(\cup_{j \in [i]} M_j)$ be the sets of uncovered vertices of A and C , respectively. Let $n_i = |V(H) \setminus V(\cup_{j \in [i]} M_j)|$ and note that $n_i \equiv r \pmod k$.

Step 1. Small matchings M_1 and M_2 covering B .

We build the first matching M_1 on vertices of $B \cup C$ of size t only if $t := \lfloor \frac{n}{k} \rfloor - |A| > 0$. Note that it is possible that $t \leq 0$ – in this case $M_1 = \emptyset$. By Proposition 3.2, we know that $t = \lfloor \frac{n}{k} \rfloor - |A| \leq \alpha n$. Since $\delta_{k-1}(H) \geq \lfloor \frac{n}{k} \rfloor$ and by the definition of t , we have $\delta_{k-1}(H[B \cup C]) \geq t$. Since $|C| \leq \lceil \frac{(k-1)n}{k} \rceil$, we have $|B| = n - |C| - |A| \geq \lfloor \frac{n}{k} \rfloor - |A| = t$. We pick arbitrary t disjoint $(k-1)$ -sets from

C . Since C is an independent set, each of the $(k-1)$ -sets has at least t neighbors in B , so we can choose a matching M_1 of size t .

Next we build the second matching M_2 that covers all the vertices in $B \setminus V(M_1)$. For each $v \in B \setminus V(M_1)$, we pick $k-2$ arbitrary vertices from C not covered by the existing matching, and an uncovered vertex $v \in V$ to complete an edge and add it to M_2 . Since $\delta_{k-1}(H) \geq \lfloor \frac{n}{k} \rfloor$ and the number of vertices covered by the existing matching is at most $k|B| \leq k\alpha n < \lfloor \frac{n}{k} \rfloor$, such an edge always exists.

Our construction guarantees that each edge in $M_1 \cup M_2$ contains at least one vertex from B and thus $|M_1 \cup M_2| \leq |B|$. We claim that $|A_1| \geq \frac{n_1-r}{k}$ and $|A_2| \geq \frac{n_2-r}{k}$. To see the bound for $|A_1|$, we separate two cases depending on t . When $t > 0$, since $|M_1| = t$, we have

$$|A_1| = \frac{n-r}{k} - t = \frac{n-r-k|M_1|}{k} = \frac{n_1-r}{k}.$$

Otherwise $t \leq 0$, we have $n_1 = n$ and $|A_1| = |A| \geq \frac{n-r}{k} = \frac{n_1-r}{k}$. For the bound for $|A_2|$, since each edge of M_2 contains at most one vertex of A , we have

$$|A_2| \geq |A_1| - |M_2| \geq \frac{n_1-r}{k} - |M_2| = \frac{n_2-r}{k}.$$

Let $s := |A_2| - \frac{n_2-r}{k} \geq 0$. Since $n_2 = n - k|M_1 \cup M_2| \geq n - k|B| \geq n - k\alpha n$ and $|C| \geq (1-\epsilon)\frac{(k-1)n}{k}$ (Proposition 3.2), we get

$$s \leq n - |C| - \frac{n - k\alpha n - r}{k} \leq \epsilon \frac{(k-1)n}{k} + \alpha n + 1 \leq 2\alpha n.$$

Step 2. A small matching M_3 .

Starting with $M_3 = \emptyset$, we will greedily add at most $2\alpha n$ edges to M_3 from $A_2 \cup C_2$ until we have $|A_3| - \frac{n_3-r}{k} \in \{0, 1\}$. Indeed, throughout the process, denote by n' the number of uncovered vertices of H and denote by A', C' the set of uncovered vertices in A, C , respectively. Let $c = |A'| - \frac{n'-r}{k}$. If $c \geq k-1$, then we arbitrarily pick $k-1$ vertices from A' and a vertex from $A' \cup C'$ to form an edge. As a result, $|A'| - \frac{n'-r}{k}$ decreases by $k-1$ or $k-2$. If $c < k-1$, then we pick c vertices from A' , $k-c-1$ vertices from C' , and form an edge with some vertex from $A' \cup C'$. In this case, $|A'| - \frac{n'-r}{k}$ decreases by c or $c-1$. The iteration stops when $|A'| - \frac{n'-r}{k}$ becomes 0 or 1 after at most $\lceil \frac{s}{k-2} \rceil \leq s \leq 2\alpha n$ steps. Note that we can always form an edge in each step because the number of covered vertices is at most $k|B| + k \cdot 2\alpha n \leq 3k\alpha n < \delta_{k-1}(H)$. So we get a matching M_3 of at most $2\alpha n$ edges.

Step 3. The last matching M_4 .

Now we have two cases, $|A_3| - \frac{n_3-r}{k} = 0$ or 1. In the first case, we will find a matching M_4 of size $|A_3|$ which leaves r vertices in C_3 . In the second case, we will find a matching M_4 of size $|A_3| - 1$ which leaves one vertex in A_3 and $r-1$ vertices in C_3 . Note that in either case we are done since $M = M_1 \cup M_2 \cup M_3 \cup M_4$ is a matching that covers all but r vertices of $V(H)$.

We define A'_3 and C'_3 as follows. If $|A_3| - \frac{n_3-r}{k} = 0$, we let $A'_3 = A_3$ and obtain C'_3 by deleting arbitrary r vertices from C_3 . Otherwise, we obtain A'_3 by deleting one arbitrary vertex from A_3 and obtain C'_3 by deleting $r-1$ arbitrary vertices

from C_3 . Note that in both cases, we have $|A'_3| - \frac{|A'_3| + |C'_3|}{k} = 0$, which implies $|C'_3| = (k-1)|A'_3|$. Furthermore, we have

$$|A'_3| \geq |A| - |M_1 \cup M_2| - |M_3| - 1 \geq \left\lfloor \frac{n}{k} \right\rfloor - \alpha n - \alpha n - 2\alpha n - 1 \geq \left\lfloor \frac{n}{k} \right\rfloor - 5\alpha n,$$

because $|M_1 \cup M_2| \leq |B| \leq \alpha n$ and $|M_3| \leq 2\alpha n$.

Let $m := |A'_3|$. Next, we partition C'_3 arbitrarily into $k-1$ parts C^1, C^2, \dots, C^{k-1} of the same size m . We want to apply Theorem 3.1 on the k -partite k -graph $H' := H[A'_3, C^1, \dots, C^{k-1}]$. Let us verify the assumptions. First, since C'_3 is independent, for any set of $k-1$ vertices v_1, \dots, v_{k-1} such that $v_i \in C^i$ for $i \in [k-1]$, the number of its non-neighbors in $A \cup B$ is at most

$$|A| + |B| - \left\lfloor \frac{n}{k} \right\rfloor \leq \frac{n}{k} + \epsilon \frac{(k-1)n}{k} - \left\lfloor \frac{n}{k} \right\rfloor \leq \epsilon n \leq 2k\epsilon m,$$

where we use (3.2) and the last inequality follows from $m = |A'_3| \geq \left\lfloor \frac{n}{k} \right\rfloor - 5\alpha n > \frac{k-1}{k^2}n$. So we have $\delta_{[k] \setminus \{1\}}(H') \geq m - 2k\epsilon m = (1 - 2k\epsilon)m$. Next, by (3.1), for any $v \in A'_3$, we have

$$\overline{\deg}_H(v, C) \leq \alpha \binom{|C|}{k-1} \leq \alpha \frac{|C|^{k-1}}{(k-1)!} \leq \alpha \frac{\left(\frac{k-1}{k}n\right)^{k-1}}{(k-1)!} \leq \alpha \frac{(km)^{k-1}}{(k-1)!} = \alpha c_k m^{k-1},$$

where $c_k = \frac{k^{k-1}}{(k-1)!}$. This implies that $\delta_{\{1\}}(H') \geq (1 - \alpha c_k)m^{k-1}$. Thus, we have

$$\delta_{\{1\}}(H')m + \delta_{[k] \setminus \{1\}}(H')m^{k-1} \geq (1 - \alpha c_k)m^{k-1}m + (1 - 2k\epsilon)mm^{k-1} > \frac{3}{2}m^k,$$

since ϵ is small enough. By Theorem 3.1, we find a perfect matching in H' , which gives the perfect matching M_4 on $H[A'_3 \cup C'_3]$. \square

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